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Sequences of (ψ, ϕ)-Weakly Contractive Mappings and Stability of Fixed Points in 2-Metric Spaces

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Abstract. The purpose of this paper is to present some new results on the stability of fixed points for certain sequences of weakly contractive mappings, known as (ψ, ϕ)-weakly contractive mappings over a variable domain in a 2-metric space. The results obtained herein extend certain known results.

1. Introduction

Recently, Barbet and Nachi [5] (see also [4]) obtained some stability results using certain new notions of convergence over a variable domain in a metric space. The above results include the earlier results of Bonsall [6] and Nadler [30]. For some useful references on stability of fixed points, we refer to [1, 3, 20, 21, 22, 32, 34, 35, 36, 37, 38, 39]. These results have been further generalized by Mishra et al.[23, 24, 25, 26, 27, 28, 29] in different settings. In this paper, we obtain a number of stability results in 2-metric spaces for a much wider class of (ψ, ϕ)-weakly contractive mappings (see Remark 1 below) which include the well known contraction mappings, nonlinear contractions due to Boyd and Wong and weakly contractive mappings (see [2, 8, 10, 18, 33, 40, 41] for details). The results obtained herein thus compliment the results of Barbet and Nachi [5] and extend the result of Mishra et al. [26] to 2-metric spaces. We note that the results so obtained are significant in the sense that 2-metric spaces differ topologically with metric spaces(see Remark 2 below).

Let (X, d) be a metric space and $T : X \to X$. Then T is called a contraction mapping if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \tag{1}$$

for all $x, y \in X$.

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$T : X \to X$ is called nonlinear contraction [8], if

$$d(Tx, Ty) \leq \alpha\left(d(x, y)\right) \tag{2}$$

for all $x, y \in X$, where $\alpha : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and $\alpha(t) < t$ for $t > 0$. We note that $\alpha(0) = 0$.

$T : X \to X$ is called weakly contractive, if

$$d(Tx, Ty) \leq d(x, y) - \phi\left(d(x, y)\right) \tag{3}$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

We note that the condition(3) follows from Tasković [40, 41]. For an earlier work in this direction, we refer to Krasnosel'skiĩ et al. [18] and Dugundji and Granas [10]. Also, that these mappings have been studied by Alber and Guerre-Delabriere [2] and Rhoades [33] as mentioned by Jachymski [17].

In this paper, we shall use the following class of mappings satisfying the so called (ψ, ϕ) condition (see for details [11, 7, 9]).

$T : X \to X$ is called (ψ, ϕ)-weakly contractive if

$$\psi\left(d(Tx, Ty)\right) \leq \psi\left(d(x, y)\right) - \phi\left(d(x, y)\right) \tag{4}$$

for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, ϕ is non-increasing and ψ is increasing(strictly).

Remark 1. It is interesting to note that if one takes $\phi(t) = (1 - k)t$, where $0 < k < 1$, then (3) reduces to (1). When $\psi(t) = t$, then condition (4) recovers condition (3). In fact, the weakly contractive mappings are also closely related to nonlinear contraction. If $\phi(t) = t - \alpha(t)$, then (3) turns into (2). Therefore

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

However, the above implication is not reversible (see [11, Example 2.2]).

This shows the generality of (ψ, ϕ)-weakly contractive mappings.

2. 2-METRIC SPACES

We first recall some basics of 2-metric spaces. For details we refer to Gähler [12] and Iséki [13, 14, 15, 16].

Definition 1. Let X be a nonempty set, consisting of at least three points. A 2-metric on X is a real-valued function ρ on $X \times X \times X$ which satisfies the following conditions:

(a) To each pair of distinct points $x, y \in X$ there exists a point $a \in X$ such that $\rho(x, y, a) \neq 0$.
(b) If at least two of x, y, a are equal then $\rho(x, y, a) = 0$.
(c) $\rho(x, y, a) = \rho(y, a, x) = \rho(x, a, y)$ for all $x, y, a \in X$.

(d) $\rho(x, y, a) \leq \rho(x, y, z) + \rho(x, z, a) + \rho(z, y, a)$ for all $x, y, z, a \in X$.

The pair (X, ρ) is called a 2-metric space. ρ is non-negative and it abstracts the properties of the area function for Euclidean triangles in the same manner as a metric abstracts the properties of the length function.

Definition 2. A sequence $\{x_n\}$ in a 2-metric space (X, ρ) is said to be convergent with limit $z \in X$ if $\lim_{n\to\infty} \rho(x_n, z, a) = 0$ for all $a \in X$. Notice that if the sequence $\{x_n\}$ converges to z, then $\lim_{n\to\infty} \rho(x_n, a, b) = \rho(z, a, b)$ for all $a, b \in X$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{m,n\to\infty} \rho(x_m, x_n, a) = 0$ for all $a \in X$. If every Cauchy sequence in X is convergent, (X, ρ) is said to be a complete 2-metric space.

Definition 3. A 2-metric space (X, ρ) is said to be bounded if there is a constant K such that $\rho(a, b, c) \leq K$ for all $a, b, c \in X$.

Remark 2. The following remarks briefly capture some distinct features of topological properties of 2-metric spaces which differ from those of metric spaces.

(i) Given any metric space which consist of more than two points, there always exists a 2-metric compatible with the topology of the space. But the converse is not always true as one can find a 2-metric space which does not have a countable basis associated with one of its arguments (see Gähler [12, Theorem 20 and Example on page 145]).
(ii) A 2-metric ρ is a continuous mapping of each of its three arguments. Generally, we cannot however assert the continuity of ρ in all the three arguments. But if it is continuous in any two arguments simultaneously, then it is continuous in all the three arguments (see Gähler [12, page 123]). So if a 2- metric is continuous in any two arguments, we shall call it continuous.
(iii) In a complete 2-metric space, a convergent sequence need not be Cauchy (see Naidu and Prasad [31, Example 0.1]).
(iv) In a 2-mertic space (X, ρ), every convergent sequence is Cauchy whenever ρ is continuous. However, the converse need not be true (see Naidu and Prasad [31, Example 0.2]).

Now we state the following analogue of conditions (1), (2), (3) and (4) for 2-metric spaces as follows.

Let (X, ρ) be a 2-metric space and $T : X \to X$. Then T is called a contraction mapping(or k -contraction) if there exists a constant $k \in (0, 1)$ such that

$$\rho(Tx, Ty, a) \leq k\rho(x, y, a) \tag{5}$$

for all $x, y, a \in X$.

It is well known that a contraction mapping on a 2-metric space X has a unique fixed point. Initially, an additional requirement of boundedness was placed on X by Iséki et al.[16] and which was dispensed with subsequently by Rhoades [32] and Lal and Singh [19] independently.

$T : X \to X$ is called a nonlinear contraction if

$$\rho(Tx, Ty, a) \leq \alpha(\rho(x, y, a)) \tag{6}$$

for all $x, y \in X$, where $\alpha : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and $\alpha(t) < t$ for $t > 0$. We note that $\alpha(0) = 0$.

$T : X \to X$ is said to be weakly contractive on X if

$$\rho(Tx, Ty, a) \leq \rho(x, y, a) - \phi(\rho(x, y, a)) \tag{7}$$

for all $x, y, a \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that $\phi(t) = 0$ if and only if $t = 0$.

$T : X \to X$ is called (ψ, ϕ)- weakly contractive if

$$\psi(\rho(Tx, Ty, a)) \leq \psi(\rho(x, y, a)) - \phi(\rho(x, y, a)) \tag{8}$$

for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, ϕ is non-increasing and ψ is increasing(strictly).

Throughout this paper, let (X, ρ) be a 2-metric space with a continuous 2-metric ρ. Let $\mathbb{N}$ be the set of naturals and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

3. STABILITY UNDER (G)-CONVERGENCE

Definition 4. [27] Let X be a 2-metric space, $\{X_n\}_{n \in \overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n \in \overline{\mathbb{N}}}$ a family of mappings. Then T_∞ is called a (G)-limit of the sequence $\{T_n\}_{n \in \mathbb{N}}$, or equivalently $\{T_n\}_{n \in \overline{\mathbb{N}}}$ satisfies the property (G) if the following condition holds:

(G): $Gr(T_\infty) \subset \liminf Gr(T_n)$: for every $z \in X_\infty$, there exists a sequence $\{x_n\}$ in $\underset{n \in \mathbb{N}}{\Pi} X_n$ such that

$$\lim_{n \to \infty} \rho(x_n, z, a) = 0 \quad \text{and} \quad \lim_{n \to \infty} \rho(T_n x_n, T_\infty z, a) = 0 \quad \text{for all } a \in X,$$

where $Gr(T)$ denotes the graph of T.

The following proposition extends a result of Barbet and Nachi [5, Proposition 1] (see also Mishra et al.[26]) for (ψ, ϕ)-weakly contractive mappings to a 2-metric space.

Proposition 1. *Let X be a 2-metric space, $\{X_n\}_{n \in \overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n \in \mathbb{N}}$ a sequence of (ψ, ϕ)-weakly contractive mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

Proof. Assume that $T_\infty : X_\infty \to X$ and $T^*_\infty : X_\infty \to X$ are (G)-limit mappings of the sequence $\{T_n\}$. Hence for any point $x \in X_\infty$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ converging to x such that $\{T_n x_n\}$ and $\{T_n y_n\}$ converge to T_∞ and T^*_∞ respectively. Therefore

$$\lim_{n\to\infty} \rho\,(T_n x_n, T_\infty x, a) = 0, \ \lim_{n\to\infty} \rho\,(\ T_n y_n, T^*_\infty x, a) = 0 \ \text{ for all } a \in X.$$

Since T_n is (ψ, ϕ)-weakly contractive for each $n \in \mathbb{N}$,

$$\psi(\rho(T_n x_n, T_n y_n, a)) \le \psi(\rho(x_n, y_n, a)) - \phi(\rho(x_n, y_n, a))$$

which implies that

$$\psi(\rho(T_n x_n, T_n y_n, a)) \le \psi(\rho(x_n, y_n, a)).$$

As ψ is increasing, from the above inequality we have

$$\rho(T_n x_n, T_n y_n, a) \le \rho(x_n, y_n, a). \tag{9}$$

By the triangular area inequality and condition (9), for all $n \in \mathbb{N}$ and for any $a \in X$, we have

$$\begin{aligned}
&\rho(T_\infty x, T^*_\infty x, a)\\
&\le \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, a)\\
&\le \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, T_n y_n)\\
&\qquad + \rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T^*_\infty x, a)\\
&\le \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, T_n y_n)\\
&\qquad + \rho(x_n, y_n, a) + \rho(T_n y_n, T^*_\infty x, a) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence we deduce that $\lim_{n\to\infty} \rho(T_\infty x, T^*_\infty x, a) = 0$ and the unicity of the limit is established. □

When $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0,1)$ in the above proposition, we get the following result.

Corollary 1 ([27, Proposition 2.2])**.** *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{\ T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of k-contraction mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

When $\psi(t) = t$ and $\phi(t) = t - \alpha(t)$ in the above proposition, the following result is obtained.

Corollary 2. [25, Proposition 3.3] *Corollary* 1 *with k-contraction replaced by nonlinear contraction.*

The following theorem is our first stability result.

Theorem 1. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) such that for all $n \in \mathbb{N}$, T_n is a (ψ,ϕ)-weakly contractive mapping where ψ is increasing and ϕ is non-increasing. If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let x_n be a fixed point of T_n for each $n \in \overline{\mathbb{N}}$. Since the property (G) holds and $x_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \quad\text{and}\quad \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \quad\text{for all}\quad a \in X.$$

We have

$$\begin{aligned}\psi\left(\rho(x_n, x_\infty, a)\right) &= \psi\left(\rho(T_n x_n, T_\infty x_\infty, a)\right)\\ &\le \psi\left(\rho(T_n x_n, T_\infty x_\infty, T_n y_n) + \rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T_\infty x_\infty, a)\right).\end{aligned}$$

Making $n \to \infty$ in the above inequality and using the continuity of ψ and ϕ and the fact that ψ is increasing and ϕ is non-increasing, we obtain

$$\begin{aligned}\lim_{n\to\infty} \psi\left(\rho(x_n, x_\infty, a)\right) &\le \lim_{n\to\infty} \psi\left(\rho(T_n x_n, T_\infty x_\infty, T_n y_n)\right.\\ &\qquad \left. + \rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T_\infty x_\infty, a)\right)\\ &= \lim_{n\to\infty} \psi\left(\rho(T_n x_n, T_n y_n, a)\right)\\ &\le \lim_{n\to\infty} \left[\psi\left(\rho(x_n, y_n, a)\right) - \phi\left(\rho(x_n, y_n, a)\right)\right]\\ &\le \lim_{n\to\infty} \left[\psi\left(\rho(x_n, y_n, x_\infty) + \rho(x_n, x_\infty, a) + \rho(x_\infty, y_n, a)\right)\right.\\ &\qquad \left. - \phi\left(\rho(x_n, y_n, x_\infty) + \rho(x_n, x_\infty, a) + \rho(x_\infty, y_n, a)\right)\right]\\ &= \lim_{n\to\infty} \psi\left(\rho(x_n, x_\infty, a)\right) - \lim_{n\to\infty} \phi\left(\rho(x_n, x_\infty, a)\right),\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \phi\left(\rho(x_n, x_\infty, a)\right) \le 0.$$

By the property of ϕ, we get $\lim_{n\to\infty} \rho(x_n, x_\infty, a) = 0$ and hence the conclusion. □

Corollary 3 ([27, Theorem 2.3])**.** *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) such that, for all $n \in \mathbb{N}$, T_n is a k-contraction. If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. It comes from Theorem 1 when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0,1)$. □

Corollary 4 ([25, Theorem 3.5])**.** *Corollary 3 with k-contraction replaced by nonlinear contraction.*

The existence of a fixed point for a (G)-limit mapping is characterized by the following result when it is a (ψ, ϕ)-weakly contractive mapping.

Theorem 2. *Let X be a 2-metric space, $\{X_n\}_{n\in \overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n \in \mathbb{N}$, T_n is a (ψ, ϕ)-weakly contractive mapping. Assume that, for any $n \in \mathbb{N}$, x_n is a fixed point of T_n. Then*

$$\begin{aligned} T_\infty \textit{ admits a fixed point } &\Leftrightarrow \{x_n\} \textit{ converges and } \lim x_n \in X_\infty \\ &\Leftrightarrow \{x_n\} \textit{ admits a subsequence converging to a point of } X_\infty. \end{aligned}$$

Proof. The necessary part is already proved in Theorem 1. To prove the sufficiency, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\lim\limits_{j\to\infty} x_{n_j} = x_\infty \in X_\infty$. By the property (G), there exists a sequence $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

Hence for any $a \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} &\rho\left(x_\infty, T_\infty x_\infty, a\right) \\ &\leq \rho(x_\infty, x_{n_j}, a) + \rho\left(T_{n_j} x_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j} x_{n_j}\right) \\ &\leq \rho(x_\infty, x_{n_j}, a) + \rho\left(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}\right) + \rho\left(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a\right) \\ &\quad + \rho\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j} x_{n_j}\right) \\ &\leq \rho\left(x_\infty, x_{n_j}, a\right) + \rho\left(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}\right) + \rho\left(x_{n_j}, y_{n_j}, a\right) \\ &\quad + \rho\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j} x_{n_j}\right) \text{ by condition (9).} \end{aligned}$$

The right hand side of the above expression tends to zero as $j \to \infty$ and hence $T_\infty x_\infty = x_\infty$, proving that x_∞ is a fixed point of T_∞. □

Corollary 5 ([25, Theorem 3.8])**.** *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n \in \mathbb{N}$, T_n is a nonlinear contraction. Assume that for any $n \in \mathbb{N}$, x_n is a fixed point of T_n. Then*

$$\begin{aligned} T_\infty \textit{ admits a fixed point } &\Leftrightarrow \{x_n\} \textit{ converges and } \lim x_n \in X_\infty \\ &\Leftrightarrow \{x_n\} \textit{ admits a subsequence converging to a point of } X_\infty. \end{aligned}$$

Remark 3. Under the assumptions of Theorem 2, and if

(i): $\liminf X_n \subset X_\infty$ (i.e., the limit of any convergent sequence $\{z_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ is in X_∞), then

$$T_\infty \text{ admits a fixed point} \Leftrightarrow \{x_n\} \text{ converges.}$$

(ii): $\limsup X_n \subset X_\infty$ (i.e., the cluster point of any sequence $\{z_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞) then

T_∞ admits a fixed point $\Leftrightarrow \{x_n\}$ admits a convergent subsequence.

The following proposition provides a sufficient condition under which a (G)-limit of a sequence of (ψ,ϕ)-weakly contractive mappings is again (ψ,ϕ)-weakly contractive.

Proposition 2. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n\in\mathbb{N}$, T_n is a (ψ,ϕ)-weakly contractive mapping. Then T_∞ is (ψ,ϕ)-weakly contractive.*

Proof. Given two points x and y in X_∞, by the property (G) there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ converging respectively to x and y such that the sequences $\{T_n x_n\}$ and $\{T_n y_n\}$ converge respectively to $T_\infty x$ and $T_\infty y$. For any $n\in\mathbb{N}$ and $a\in X$,

$$\begin{aligned}
&\psi\left(\rho(T_\infty x, T_\infty y, a)\right) \\
&\le \psi\left(\rho(T_\infty x, T_\infty y, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty y, a)\right) \\
&\le \psi\left(\rho(T_\infty x, T_\infty y, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty y, T_n y_n)\right. \\
&\qquad \left. + \rho\left(T_n x_n, T_n y_n, a\right) + \rho(T_n y_n, T_\infty y, a)\right).
\end{aligned}$$

Letting $n\to\infty$, and using the continuity of both ψ and ϕ we have

$$\begin{aligned}
\psi\left(\rho(T_\infty x, T_\infty y, a)\right) &\le \lim_{n\to\infty} \psi\left(\rho(T_n x_n, T_n y_n, a)\right) \\
&\le \lim_{n\to\infty} \left[\psi\left(\rho(x_n, y_n, a)\right) - \phi\left(\rho(x_n, y_n, a)\right)\right].
\end{aligned}$$

Hence we conclude that $\psi\left(\rho(T_\infty x, T_\infty y, a)\right) \le \psi\left(\rho(x,y,a)\right) - \phi\left(\rho(x,y,a)\right)$. □

Corollary 6. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n\in\mathbb{N}$, T_n is a k-contraction from X_n to X. Then T_∞ is a k-contraction.*

Proof. This comes from Proposition 2 when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k\in(0,1)$. □

Corollary 7 ([25, Proposition 3.10])**.** *Corollary* 6 *with k-contraction replaced by nonlinear contraction.*

Under a compactness assumption, the existence of a fixed point of the (G)-limit mapping can be obtained from the existence of fixed points of the (ψ,ϕ)-weakly contractive mappings T_n.

Theorem 3. *Let* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *be a family of nonempty subsets of a 2-metric space* X *and* $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *a family of mappings satisfying the property (G) and such that, for any* $n \in \mathbb{N}$, T_n *is a* (ψ, ϕ)*-weakly contractive mapping where* ψ *is increasing and* ϕ *is non-increasing. Assume that* $\limsup X_n \subset X_\infty$ *and* $\bigcup\limits_{n\in\mathbb{N}} X_n$ *is relatively compact. If for any* $n \in \mathbb{N}$, T_n *admits a fixed point* x_n, *then the (G)-limit mapping* T_∞ *admits a fixed point* x_∞ *and the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x_∞.

Proof. Let x_n be the fixed point of T_n for $n \in \mathbb{N}$. From compactness condition, there exists a convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$. Now by Remark 3, T_∞ admits a fixed point x_∞ and by Theorem 1, the sequence $\{x_n\}$ converges to x_∞. □

Corollary 8 ([27, Theorem 2.10])**.** *Let* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *be a family of nonempty subsets of a 2-metric space* X *and* $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *a family of mappings satisfying the property (G) and such that, for any* $n \in \mathbb{N}$, T_n *is a* k*-contraction. Assume that* $\limsup X_n \subset X_\infty$ *and* $\bigcup\limits_{n\in\mathbb{N}} X_n$ *is relatively compact. If for any* $n \in \mathbb{N}$, T_n *admits a fixed point* x_n, *then the (G)-limit mapping* T_∞ *admits a fixed point* x_∞ *and the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x_∞.

Proof. This comes from Theorem 3, when $\psi(t) = t$ and $\phi(t) = (1 - k)t$ and $k \in (0, 1)$. □

Corollary 9 ([25, Theorem 3.12])**.** *Corollary* 8 *with* k*-contraction replaced by nonlinear contraction.*

Proof. This comes from Theorem 3, when $\psi(t) = t$ and $\phi(t) = t - \alpha(t)$. □

The following notion of convergence is weaker than (G)-convergence and has been studied in [27].

Definition 5. [27] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings. Then T_∞ is called a (G$^-$) limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$, or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G$^-$), if the following condition holds:

(G$^-$): $Gr(T_\infty) \subset \limsup Gr(T_n)$: for all $z \in X_\infty$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\prod\limits_{n\in\mathbb{N}} X_n$, which has a subsequence $\{x_{n_j}\}$ such that

$$\lim_{j\to\infty} \rho(x_{n_j}, z, a) = 0 \text{ and } \lim_{j\to\infty} \rho(T_{n_j}x_{n_j}, T_\infty z, a) = 0, \text{ for all } a \in X.$$

The following result establishes that a fixed point of a (G$^-$)-limit mapping is a cluster point of the sequence of fixed points associated with $\{T_n\}$.

Theorem 4. *Let $\{X_n\}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of (ψ,ϕ)-weakly contractive mappings satisfying the property (G^-), where ψ is increasing and ϕ is non-increasing. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. By the property (G^-), there exists a sequence $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ which has a subsequence $\{y_{n_j}\}$ such that

$$\lim_{j\to\infty} \rho(y_{n_j}, x_\infty, a) = 0 \text{ and } \lim_{j\to\infty} \rho\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right) = 0 \text{ for all } a \in X.$$

By the triangular area inequality, we have

$$\begin{aligned}\rho(x_{n_j}, x_\infty, a) \le\ & \rho(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}) \\ & + \rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a) + \rho(T_{n_j} y_{n_j}, T_\infty x_\infty, a).\end{aligned}$$

Since ψ is increasing,

$$\begin{aligned}\psi\left(\rho(x_{n_j}, x_\infty, a)\right) \le\ & \psi\left(\rho(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j})\right. \\ & \left. + \rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a) + \rho(T_{n_j} y_{n_j}, T_\infty x_\infty, a)\right).\end{aligned}$$

Now using the properties of ψ and ϕ and making $j \to \infty$, we get

$$\begin{aligned}& \lim_{j\to\infty} \psi\left(\rho(x_{n_j}, x_\infty, a)\right) \\ & \le \lim_{j\to\infty} \psi\left(\rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a)\right) \\ & \le \lim_{j\to\infty} \left[\psi\left(\rho(x_{n_j}, y_{n_j}, a)\right) - \phi\left(\rho(x_{n_j}, y_{n_j}, a)\right)\right] \\ & \le \lim_{j\to\infty} \left[\psi\left(\rho(x_{n_j}, y_{n_j}, x_\infty) + \rho(x_{n_j}, x_\infty, a) + \rho(x_\infty, y_{n_j}, a)\right)\right. \\ & \qquad \left. - \phi\left(\rho(x_{n_j}, y_{n_j}, x_\infty) + \rho(x_{n_j}, x_\infty, a) + \rho(x_\infty, y_{n_j}, a)\right)\right] \\ & = \lim_{j\to\infty} \psi\left(\rho(x_{n_j}, x_\infty, a)\right) - \lim_{j\to\infty} \phi\left(\rho(x_{n_j}, x_\infty, a)\right).\end{aligned}$$

Hence

$$\lim_{j\to\infty} \phi\left(\rho(x_{n_j}, x_\infty, a)\right) \le 0.$$

By the property of ϕ, we deduce that

$$\lim_{j\to\infty} \rho(x_{n_j}, x_\infty, a) = 0.$$

Thus $\{x_{n_j}\}$ converges to x_∞, the fixed point of T_∞. □

Corollary 10 ([27, Theorem 2.12])**.** *Let $\{X_n\}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of k-contraction mappings satisfying the property (G^-). If for any $n \in \mathbb{N}$, x_n is a fixed point of T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. This comes from Theorem 4, when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0,1)$. $\square$

Corollary 11 ([25, Theorem 3.15]). *Corollary 10 with k-contraction replaced by nonlinear contraction.*

4. Stability under (H)-convergence

Definition 6. [27] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings. Then T_∞ is called an (H)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H) if the following condition holds:

(H): For all sequences $\{x_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that

$$\lim_{n\to\infty} \rho(x_n, y_n, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n x_n, T_\infty y_n, a) = 0 \text{ for all } a \in X.$$

The following theorem is our second stability result using the (H)-convergence in 2-metric spaces.

Theorem 5. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X, $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that T_∞ is a (ψ,ϕ)-weakly contractive mapping. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By the property (H), there exists a sequence $\{y_n\}$ in X_∞ such that $\lim\limits_{n\to\infty} \rho(x_n, y_n, a) = 0$ and $\lim\limits_{n\to\infty} \rho(T_n x_n, T_\infty y_n, a) = 0$ for any $a \in X$. Hence for any $a \in X$,

$$\begin{aligned}
\rho(x_n, x_\infty, a) &= \rho(T_n x_n, T_\infty x_\infty, a) \\
&\le \rho(T_n x_n, T_\infty y_n, a) \\
&\quad + \rho(T_\infty y_n, T_\infty x_\infty, a) + \rho(T_n x_n, T_\infty x_\infty, T_\infty y_n).
\end{aligned}$$

Since ψ is increasing,

$$\begin{aligned}
\psi\left(\rho(x_n, x_\infty, a)\right) \le \psi\,(&\rho(T_n x_n, T_\infty y_n, a) \\
&+ \rho(T_\infty y_n, T_\infty x_\infty, a) + \rho(T_n x_n, T_\infty x_\infty, T_\infty y_n)).
\end{aligned}$$

Now using the properties of ψ and ϕ and making $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \psi\left(\rho(x_n, x_\infty, a)\right) &\le \lim_{n\to\infty} \psi\left(\rho(T_\infty y_n, T_\infty x_\infty, a)\right) \\
&\le \lim_{n\to\infty} \left[\psi\left(\rho(y_n, x_\infty, a)\right) - \phi\left(\rho(y_n, x_\infty, a)\right)\right] \\
&\le \lim_{n\to\infty} \left[\psi\left(\rho(y_n, x_\infty, x_n) + \rho(y_n, x_n, a) + \rho(x_n, x_\infty, a)\right)\right. \\
&\qquad \left. - \phi\left(\rho(y_n, x_\infty, x_n) + \rho(y_n, x_n, a) + \rho(x_n, x_\infty, a)\right)\right] \\
&= \lim_{n\to\infty} \psi\left(\rho(x_n, x_\infty, a)\right) - \lim_{n\to\infty} \phi\left(\rho(x_n, x_\infty, a)\right).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \phi\left(\rho(x_n, x_\infty, a)\right) \le 0.$$

By the property of ϕ, we deduce that

$$\lim_{n\to\infty} \rho(x_n, x_\infty, a) = 0,$$

and the conclusion follows. □

Corollary 12 ([27, Theorem 3.4])**.** *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of nonempty subsets of X, $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings satisfying the property (H) and such that T_∞ is a k-contraction. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Theorem 5, when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0, 1)$. □

Corollary 13 ([25, Theorem 4.5])**.** *Corollary* 12 *with k-contraction replaced by nonlinear contraction.*

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